Математика 2, дополнителни вежби
07.03.2022 и 08.03.2022

1. Да се пресметаат следните интеграли:

а) $\displaystyle\int \frac{dx}{4+7\sin^2 x}$, б) $\displaystyle\int \frac{dx}{5\cos^2 x - 3}$, в) $\displaystyle\int \frac{dx}{5\cos^2 x - 8}$,

г) $\displaystyle\int \frac{dx}{10\sin^2 x - 3}$, д) $\displaystyle\int \frac{dx}{10\sin^2 x - 13}$

Решение: Интеграли од типот $\int R(\operatorname{tg} x)\,dx$

смена $\operatorname{tg} x = t$ $\quad (x = \operatorname{arctg} x)$

$$\frac{dx}{\cos^2 x} = dt \quad \left(dx = \frac{dt}{t^2+1} \right)$$

а) $$\int \frac{dx}{4+7\sin^2 x} = \int \frac{dx}{4(\sin^2 x + \cos^2 x) + 7\sin^2 x}$$

$$= \int \frac{dx}{11\sin^2 x + 4\cos^2 x} = \int \frac{dt}{\left(11\underbrace{\frac{\sin^2 x}{\cos^2 x}}_{=\operatorname{tg}^2 t} + 4\right)\cdot \cos^2 x}$$

$$= \left| \begin{matrix} \text{смена:} \\ \operatorname{tg} x = t, \quad \dfrac{dx}{\cos^2 x} = dt \end{matrix} \right|$$

$$= \int \frac{dt}{11t^2+4} = \frac{1}{11}\int \frac{dt}{t^2 + \underbrace{\left(\frac{2}{\sqrt{11}}\right)^2}_{=a^2}}$$

$$= \frac{1}{\underbrace{11}_{=(\sqrt{11})^2}} \cdot \frac{1}{\frac{2}{\sqrt{11}}} \cdot \operatorname{arctg} \frac{t}{\frac{2}{\sqrt{11}}} + C$$

$$= \frac{1}{2\sqrt{11}} \operatorname{arctg} \frac{\sqrt{11}}{2} t + C = \frac{1}{2\sqrt{11}} \operatorname{arctg}\left(\frac{\sqrt{11}}{2} \operatorname{tg} x\right) + C$$

б) $$\int \frac{dx}{5\cos^2 x - 3} = \int \frac{dx}{5\cos^2 x - 3(\sin^2 x + \cos^2 x)}$$

$$= \int \frac{dx}{2\cos^2 x - 3\sin^2 x} = -\int \frac{dx}{3\sin^2 x - 2\cos^2 x}$$

$$= -\int \frac{dx}{\left(3\frac{\sin^2 x}{\cos^2 x} - 2\right)\cos^2 x} = \left| \begin{array}{l} \text{смена:} \\ \operatorname{tg} x = t \\ \frac{dx}{\cos^2 x} = dt \end{array} \right|$$

$$= -\int \frac{dt}{3t^2 - 2} = -\frac{1}{3} \int \frac{dt}{t^2 - \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2}$$

$$= -\frac{1}{\cancel{3}_{\sqrt{3}}} \cdot \frac{1}{2 \cdot \frac{\sqrt{2}}{\cancel{\sqrt{3}}}} \ln \left| \frac{t - \frac{\sqrt{2}}{\sqrt{3}}}{t + \frac{\sqrt{2}}{\sqrt{3}}} \right| + C$$

$$= \sqrt{2} \cdot \sqrt{3} = \sqrt{6}$$

$$= -\frac{1}{2\sqrt{6}} \cdot \ln\left| \frac{\frac{\sqrt{3}t - \sqrt{2}}{\sqrt{3}}}{\frac{\sqrt{3}t + \sqrt{2}}{\sqrt{3}}} \right| + C$$

$$= -\frac{1}{2\sqrt{6}} \cdot \ln\left| \frac{\sqrt{3}t - \sqrt{2}}{\sqrt{3}t + \sqrt{2}} \right| + C$$

$$= -\frac{1}{2\sqrt{6}} \ln\left| \frac{\sqrt{3}\,\mathrm{tg}\,x - \sqrt{2}}{\sqrt{3}\,\mathrm{tg}\,x + \sqrt{2}} \right| + C$$

б) $$\int \frac{dx}{5\cos^2 x - 8} = \int \frac{dx}{5\cos^2 x - 8(\sin^2 x + \cos^2 x)}$$

$$= \int \frac{dx}{-3\cos^2 x - 8\sin^2 x} = -\int \frac{dx}{8\sin^2 x + 3\cos^2 x}$$

$$= -\int \frac{dx}{\Big(8\underbrace{\frac{\sin^2 x}{\cos^2 x}}_{\mathrm{tg}^2 x} + 3\Big)\cos^2 x} = \left| \begin{array}{l} \text{Смена:} \\ \mathrm{tg}\,x = t \\ \frac{dx}{\cos^2 x} = dt \end{array} \right|$$

$$= -\int \frac{dt}{8t^2 + 3} = -\frac{1}{8} \int \frac{dt}{t^2 + \frac{3}{8}} \longrightarrow = \left(\frac{\sqrt{3}}{\sqrt{8}}\right)^2$$

$$= -\frac{1}{\cancel{8}_{\sqrt{8}}} \cdot \frac{1}{\left(\frac{\sqrt{3}}{\cancel{\sqrt{8}}}\right)} \cdot \mathrm{arctg}\,\frac{t}{\frac{\sqrt{3}}{\sqrt{8}}} + C$$

$$= -\frac{1}{\sqrt{24}} \operatorname{arctg} \frac{\sqrt{8}\cdot t}{\sqrt{3}} + C$$

$$\hookrightarrow \sqrt{6\cdot 4} = \sqrt{6}\cdot\sqrt{4} = 2\sqrt{6}$$

$$= -\frac{1}{2\sqrt{6}} \operatorname{arctg}\left(\frac{\sqrt{8}}{\sqrt{3}} \operatorname{tg} x\right) + C$$

i) $\displaystyle\int \frac{dx}{10\sin^2 x - 3} = \int \frac{dx}{10\sin^2 x - 3\cdot(\sin^2 x + \cos^2 x)}$

$$= \int \frac{dx}{7\sin^2 x - 3\cos^2 x} = \int \frac{dx}{\left(7\cdot\frac{\sin^2 x}{\cos^2 x} - 3\right)\cos^2 x}$$

$$= \left| \begin{matrix} \text{смена: } \operatorname{tg} x = t \\ \frac{dx}{\cos^2 x} = dt \end{matrix} \right| = \int \frac{dt}{7t^2 - 3} =$$

$$= \frac{1}{7}\int \frac{dt}{t^2 - \frac{3}{7}} = \frac{1}{7}\int \frac{1}{t^2 - \Big(\underbrace{\frac{\sqrt{3}}{\sqrt{7}}}_{=a}\Big)^2} dt$$

$$= \frac{1}{\cancel{7}_{\sqrt{7}}} \cdot \frac{1}{2\cdot\frac{\sqrt{3}}{\cancel{\sqrt{7}}}} \ln\left|\frac{t - \frac{\sqrt{3}}{\sqrt{7}}}{t + \frac{\sqrt{3}}{\sqrt{7}}}\right| + C$$

$$= \frac{1}{2\sqrt{21}} \ln \left| \frac{\sqrt{7} \cdot t - \sqrt{3}}{\sqrt{7} \cdot t + \sqrt{3}} \right| + C$$

$$= \frac{1}{2\sqrt{21}} \ln \left| \frac{\sqrt{7}\, tg\, x - \sqrt{3}}{\sqrt{7}\, tg\, x + \sqrt{3}} \right| + C$$

g) $$\int \frac{dx}{10\sin^2 x - 13} = \int \frac{dx}{10\sin^2 x - 13(\sin^2 x + \cos^2 x)}$$

$$= \int \frac{dx}{-3\sin^2 x - 13\cos^2 x} = -\int \frac{dx}{3\sin^2 x + 13\cos^2 x}$$

$$= -\int \frac{dx}{\left(3 \cdot \frac{\sin^2 x}{\cos^2 x} + 13\right) \cdot \cos^2 x} = \left| \begin{array}{l} \text{Смена: } tg\, x = t \\ \frac{dx}{\cos^2 x} = dt \end{array} \right|$$

$$= -\int \frac{dt}{3t^2 + 13} = -\frac{1}{3} \int \frac{dt}{t^2 + \frac{13}{3}}$$

$$\frac{13}{3} \rightarrow \left(\frac{\sqrt{13}}{\sqrt{3}}\right)^2$$

$$= -\frac{1}{\underset{\sqrt{3}}{\cancel{3}}} \cdot \frac{1}{\frac{\sqrt{13}}{\cancel{\sqrt{3}}}} \operatorname{arctg} \frac{t}{\frac{\sqrt{13}}{\sqrt{3}}} + C$$

$$= -\frac{1}{\sqrt{39}} \operatorname{arctg}\left(\frac{\sqrt{3}}{\sqrt{13}} \cdot t\right) + C = -\frac{1}{\sqrt{39}} \operatorname{arctg}\left(\frac{\sqrt{3}}{\sqrt{13}} tg\, x\right) + C$$

2. Да се пресметаат следните интеграли:

а) $\int \frac{e^{6x}}{e^x - 2}\,dx$ б) $\int \frac{2x+1}{e^x}\,dx$

Решение: $\int R(a^x)\,dx$, $a = const > 0$

$$a^x = t \,, \quad a^x \ln a \, dx = dt$$

$a = e$

$$e^x = t \,, \quad e^x dx = dt$$

$$\Downarrow$$

$$x = \ln t \,, \quad dx = \frac{dt}{t}$$

$(e^x)^5 \cdot e^x$

а) $\int \frac{e^{6x}}{e^x - 2}\,dx = \int \frac{(e^x)^6}{e^x - 2}\,dx = \left| \begin{array}{l} \text{смена } e^x = t \\ e^x dx = dt \end{array} \right|$

$$= \int \frac{t^5}{t-2} \cdot dt = (*)$$

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + a \cdot b^{n-2} + b^{n-1})$

$a = t,\ b = 2,\ n = 5$

$$(*) = \int \frac{t^5 - 2^5 + 2^5}{t-2}\,dt = \int \left[\frac{t^5 - 2^5}{t-2} + \frac{2^5}{t-2} \right] dt$$

$$= \int \frac{\cancel{(t-2)}\left(t^4+t^3\cdot 2+t^2\cdot 2^2+t\cdot 2^3+2^4\right)}{\cancel{t-2}}\,dt + 32\int\frac{dt}{t-2}$$

$$= \int \left(t^4+2t^3+4t^2+8t+16\right)dt + 32\ln|t-2|$$

$$= \frac{t^5}{5} + \cancel{2}\,\frac{t^4}{\cancel{4}_{\,2}} + 4\cdot\frac{t^3}{3} + \cancel{8}^{\,4}\,\frac{t^2}{\cancel{2}} + 16t + 32\ln|t-2| + C$$

$$= \frac{1}{5}t^5 + \frac{1}{2}t^4 + \frac{4}{3}\cdot t^3 + 4t^2 + 16t + 32\ln|t-2| + C$$

$$= \frac{1}{5}e^{5x} + \frac{1}{2}e^{4x} + \frac{4}{3}e^{3x} + 4e^{2x} + 16e^{x} + 32\ln|e^{x}-2| + C$$

б) $$\int \frac{2x+1}{e^x}\,dx = \left| \begin{array}{l} \text{смена: } e^x=t \\ x=\ln t \\ dx=\frac{dt}{t} \end{array} \right| = \int \frac{2\ln t+1}{t}\cdot\frac{dt}{t}$$

$$= \int \frac{2\ln t+1}{t^2}\,dt = \int\left(2\,\frac{\ln t}{t^2} + \frac{1}{t^2}\right)dt$$

$$= 2\int t^{-2}\ln t + \int t^{-2}\,dt = 2\int t^{-2}\ln t + \frac{t^{-2+1}}{-2+1}$$

$$= 2\int t^{-2}\ln t - \frac{1}{t} = \left| \begin{array}{ll} \text{П.И:} & \\ u=\ln t & dv=t^{-2}\,dt \\ du=\frac{dt}{t} & v=\int t^{-2}\,dt=-\frac{1}{t} \end{array} \right|$$

$$= 2\left(-\frac{1}{t}\ln t - \int\left(-\frac{1}{t}\right)\cdot\frac{dt}{t}\right) - \frac{1}{t}$$

$$= -2\left(-\frac{1}{t}\ln t + \int\frac{dt}{t^2}\right) - \frac{1}{t} = 2\left(-\frac{1}{t}\ln t - \frac{1}{t}\right) - \frac{1}{t} + C$$

$$= -2\cdot\frac{1}{t}\ln t \underbrace{- \frac{2}{t} - \frac{1}{t}}_{-\frac{3}{t}} + C = -\frac{1}{t}\left(2\ln t + 3\right) + C$$

$$= -\frac{1}{e^x}\left(2\underbrace{\ln e^x}_{=x} + 3\right) + C = -\frac{2x+3}{e^x} + C$$

алтернативен начин:

$$\int\frac{2x+1}{e^x}\,dx = \int(2x+1)e^{-x}\,dx = \left|\begin{array}{l} \text{смена: } -x = t \\ x = -t \\ dx = -dt \end{array}\right|$$

$$= \int(-2t+1)e^t\cdot(-dt) = \int(2t-1)e^t\,dt$$

$$= \left|\begin{array}{ll} \text{П.И:} & \\ u = 2t-1 & dv = e^t dt \\ du = 2dt & v = \int e^t dt = e^t \end{array}\right|$$

$$= (2t-1)e^t - \int 2e^t dt = (2t-1)e^t - 2\int e^t dt$$

$$= (2t-1)e^t - 2e^t + C = (2t-3)e^t + C$$

$$= (2\cdot(-x)-3)e^{-x} + C = -(2x+3)e^{-x} + C = -\frac{2x+3}{e^x} + C$$